Chapter 8: Continuity

EXERCISE 8.1 [PAGES 111 - 112]

Exercise 8.1 | Q 1.1 | Page 111

Examine the continuity of $f(x) = x^3 + 2x^2 - x - 2$ at $x = -2$.

SOLUTION

$f(x) = x^3 + 2x^2 - x - 2$

Here f(x) is a polynomial function and hence

it is continuous for all $x \in R$.

$\therefore$ f(x) is continuous at $x = -2$.

Exercise 8.1 | Q 1.2 | Page 111

Examine the continuity of $f(x) = \dfrac{x^2 - 9}{x - 3}$ on R.

SOLUTION

$f(x) = \dfrac{x^2 - 9}{x - 3}; x \in R$

f(x) is a rational function and is continuous for all $x \in R$, except at the points where denominator becomes zero. Here, denominator $x - 3 = 0$ when $x = 3$.

$\therefore$ Function f is continuous for all $x \in R$, except at $x = 3$, where it is not defined.

Exercise 8.1 | Q 2.1 | Page 111

Examine whether the function is continuous at the points indicated against them.

$f(x) = x^3 - 2x + 1$, for $x \le 2$
$= 3x - 2$, for $x > 2$, at $x = 2$.

SOLUTION

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} \left(x^3 - 2x + 1\right)$$

$= = (2)^3 - 2(2) + 1 = 5$

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (3x - 2)$$

= 3(2) - 2 = 4

$$\therefore \lim_{x \to 2^-} f(x) \neq \lim_{x \to 2^+} f(x)$$

∴ Function f is discontinuous at x = 2

Exercise 8.1 | Q 2.2 | Page 111

Examine whether the function is continuous at the points indicated against them.

$$f(x) = \frac{x^2 + 18x - 19}{x - 1} \qquad \text{for } x \neq 1$$

$$= 20 \qquad \text{for } x = 1, \text{ at } x = 1$$

SOLUTION

$$\lim_{x \to 1} f(x) = \lim_{x \to 1} \frac{x^2 + 18x - 19}{x - 1}$$

$$= \lim_{x \to 1} \frac{x^2 + 19x - x - 19}{x - 1}$$

$$= \lim_{x \to 1} \frac{x(x + 19) - 1(x + 19)}{x - 1}$$

$$= \lim_{x \to 1} \frac{(x - 1)(x + 19)}{(x - 1)}$$

$$= \lim_{x \to 1} (x + 19) \quad ....[\because x \to 1, \therefore x \neq 1, \therefore x - 1 \neq 0]$$

= 1 + 19 = 20

Also, f(1) = 20

$$\therefore \lim_{x \to 1} f(x) = f(1)$$

∴ f(x) is continuous at x = 1

Exercise 8.1 | Q 3.1 | Page 112

Test the continuity of the following function at the points indicated against them.

$$f(x) = \frac{\sqrt{x-1}-(x-1)^{\frac{1}{3}}}{x-2} \quad \text{for } x \neq 2$$

$= \frac{1}{5}$ for x = 2, at x = 2

SOLUTION

$f(2) = \frac{1}{5}$ ......(given)

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{\sqrt{x-1}-(x-1)^{\frac{1}{3}}}{x-2}$$

Put x - 1 = y

∴ x = 1 + y

∴ As x → 2, y → 1

$$\therefore \lim_{x\to 2} f(x) = \lim_{y\to 1} \frac{\sqrt{y}-y^{\frac{1}{3}}}{1+y-2}$$

$$= \lim_{y\to 1} \frac{y^{\frac{1}{2}}-1-y^{\frac{1}{3}}+1}{y-1}$$

$$= \lim_{y\to 1} \frac{\left(y^{\frac{1}{2}}-1\right)-\left(y^{\frac{1}{3}}-1\right)}{y-1}$$

$$= \lim_{y\to 1} \left(\frac{y^{\frac{1}{2}}-1}{y-1}-\frac{y^{\frac{1}{3}}-1}{y-1}\right)$$

$$= \lim_{y \to 1} \frac{y^{\frac{1}{2}} - 1^{\frac{1}{2}}}{y - 1} - \lim_{y \to 1} \frac{y^{\frac{1}{3}} - 1^{\frac{1}{3}}}{y - 1}$$

$$= \frac{1}{2}(1)^{\frac{-1}{2}} - \frac{1}{3}(1)^{\frac{-2}{3}} \quad ....[\because \lim_{x \to a} \frac{x^n - a^n}{x - a} = n.a^{\text{n-1}}]$$

$$= \frac{1}{2} - \frac{1}{3}$$

$$= \frac{1}{6}$$

$$\therefore \lim_{x \to 2} f(x) \neq f(2)$$

∴ f(x) is discontinuous at x = 2

Exercise 8.1 | Q 3.2 | Page 112

Test the continuity of the following function at the points indicated against them.

$$f(x) = \frac{x^3 - 8}{\sqrt{x+2} - \sqrt{3x-2}} \quad \text{for } x \neq 2$$

= -24 for x = 2, at x = 2

SOLUTION

f(2) = -24 ....(given)

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{x^3 - 8}{\sqrt{x+2} - \sqrt{3x-2}}$$

$$= \lim_{x\to 2} \frac{x^3-8}{\sqrt{x+2}-\sqrt{3x-2}} \times \frac{\sqrt{x+2}+\sqrt{3x-2}}{\sqrt{x+2}+\sqrt{3x-2}}$$

$$= \lim_{x\to 2} \frac{(x^3-8)\left(\sqrt{x+2}+\sqrt{3x-2}\right)}{(x+2)-(3x-2)}$$

$$= \lim_{x\to 2} \frac{(x^3-2^3)\left(\sqrt{x+2}+\sqrt{3x-2}\right)}{-2x+4}$$

$$= \lim_{x\to 2} \frac{(x-2)(x^2+2x+4)\left(\sqrt{x+2}+\sqrt{3x-2}\right)}{-2(x-2)}$$

$$= \lim_{x\to 2} \frac{(x^2+2x+4)\left(\sqrt{x+2}+\sqrt{3x-2}\right)}{-2}$$ ....[∵ x→2, x ≠ 2 ∴ x- 2 ≠ 0]

$$= \frac{-1}{2} \lim_{x\to 2} (x^2+2x+4)\left(\sqrt{x+2}+\sqrt{3x-2}\right)$$

$$= \frac{-1}{2} \lim_{x\to 2} (x^2+2x+4) \lim_{x\to 2} \left(\sqrt{x+2}+\sqrt{3x-2}\right)$$

$$= \frac{-1}{2} \times [2^2+2(2)+4] \times \left(\sqrt{2+2}+\sqrt{3(2)-2}\right)$$

$$= \frac{-1}{2} \times 12 \times (2+2)$$

= -24

$\therefore \lim_{x\to 2} f(x) = f(2)$

∴ f(x) is continuous at x = 2

Exercise 8.1 | Q 3.3 | Page 112

Test the continuity of the following function at the points indicated against them.

f(x) = 4x + 1, for x ≤ 3

$= \frac{59-9x}{3}$, for x > 3 at x = $\frac{8}{3}$.

SOLUTION

$$\lim_{x\to\left(\frac{8}{3}\right)^-} f(x) = \lim_{x\to\left(\frac{8}{3}\right)^-} (4x+1)$$

$= 4\left(\frac{8}{3}\right) + 1$

$= \frac{32}{3} + 1$

$= \frac{35}{3}$

$$\lim_{x\to\left(\frac{8}{3}\right)^+} f(x) = \lim_{x\to\left(\frac{8}{3}\right)^+} \frac{59-9x}{3}$$

$= \frac{59-9\left(\frac{8}{3}\right)}{3}$

$= \frac{59-24}{3}$

$= \frac{35}{3}$

f(x) = 4x + 1, $\quad x \le \left(\frac{8}{3}\right)$

$\therefore f\left(\frac{8}{3}\right) = 4\left(\frac{8}{3}\right) + 1$

$= \frac{32}{3} + 1$

$= \frac{35}{3}$

$$\lim_{x\to\left(\frac{8}{3}\right)^-} f(x) = \lim_{x\to\left(\frac{8}{3}\right)^+} f(x) = f\left(\frac{8}{3}\right)$$

$\therefore$ f(x) is continuous at x = $\frac{8}{3}$

Exercise 8.1 | Q 3.4 | Page 112

Test the continuity of the following function at the points indicated against them.

$f(x) = \frac{x^3 - 27}{x^2 - 9}$ for 0 ≤ x <3

$= \frac{9}{2}$ for 3 ≤ x ≤ 6

at x = 3

SOLUTION

f(3) = $\frac{9}{2}$ ....(given)

$\lim_{x \to 3} f(x) = \lim_{x \to 3} \frac{x^3 - 27}{x^2 - 9}$

$= \lim_{x \to 3} \frac{(x-3)\left(x^2 + 3x + 9\right)}{(x-3)(x+3)}$

$= \lim_{x \to 3} \frac{x^2 + 3x + 9}{x + 3}$ [As x → 3, x ≠ 3 ∴ x - 3 ≠ 0]

$= \frac{(3)^2 + 3(3) + 9}{3 + 3}$

$= \frac{9 + 9 + 9}{6}$

$= \frac{27}{6}$

$= \frac{9}{2}$

$\therefore \lim_{x \to 3} f(x) = f(3)$

∴ Function f is continuous at x = 3

Exercise 8.1 | Q 4.1 | Page 112

If f(x) = $\frac{24^x - 8^x - 3^x + 1}{12^x - 4^x - 3^x + 1}$ for x ≠ 0

= k, for x = 0 is continuous at x = 0, find k.

SOLUTION

Function f is continuous at x = 0

$\therefore$ f(0) = $\lim\limits_{x \to 0} f(x)$

$\therefore$ k = $\lim\limits_{x \to 0} \frac{24^x - 8^x - 3^x + 1}{12^x - 4^x - 3^x + 1}$

= $\lim\limits_{x \to 0} \frac{8^x.3^x - 8^x - 3^x + 1}{4^x.3^x - 4^x - 3^x + 1}$

= $\lim\limits_{x \to 0} \frac{8^x(3^x - 1) - 1(3^x - 1)}{4^x(3^x - 1) - 1(3^x - 1)}$

= $\lim\limits_{x \to 0} \frac{(3^x - 1)(8^x - 1)}{(3^x - 1)(4^x - 1)}$

= $\lim\limits_{x \to 0} \frac{8^x - 1}{4^x - 1}$ $\left[\left[\because x \to 0, 3^x \to 3^0\right], \left[\therefore 3^x \to 1 \therefore 3^x \neq 1\right], \left[\therefore 3^x - 1 \neq 0\right]\right]$

= $\lim\limits_{x \to 0} \left(\frac{\frac{8^x - 1}{x}}{\frac{4^x - 1}{x}}\right)$ .....[$\because$ x → 0, $\therefore$ x ≠ 0]

= $\frac{\log 8}{\log 4}$ ....$\left[\because \lim\limits_{x \to 0} \left(\frac{a^x - 1}{x}\right) = \log a\right]$

= $\frac{\log(2)^3}{\log(2)^2}$

= $\frac{3 \log 2}{3 \log 2}$

$\therefore$ f(0) = $\frac{3}{2}$

Exercise 8.1 | Q 4.2 | Page 112

If $f(x) = \frac{5^x + 5^{-x} - 2}{x^2}$ for x ≠ 0

= k for x = 0 is continuous at x = 0, find k

SOLUTION

Function f is continuous at x = 0

$$\therefore f(0) = \lim_{x \to 0} f(x)$$

$$\therefore \text{k} = \lim_{x \to 0} \frac{5^x + 5^{-x} - 2}{x^2}$$

$$= \lim_{x \to 0} \frac{5^x + \frac{1}{5^x} - 2}{x^2}$$

$$= \lim_{x \to 0} \frac{(5^x)^2 + 1 - 2(5^x)}{5^x . x^2}$$

$$= \lim_{x \to 0} \frac{(5^x - 1)^2}{5^x . x^2} \quad .....[\because a^2 - 2ab + b^2 = (a - b)^2]$$

$$= \lim_{x \to 0} \left(\frac{5^x - 1}{x}\right)^2 . \frac{1}{5^x}$$

$$= \lim_{x \to 0} \left(\frac{5^x - 1}{x}\right)^2 \times \lim_{x \to 0} \frac{1}{5^x}$$

$$= (\log 5)^2 \times \frac{1}{5^0} \quad .... \left[\because \lim_{x \to 0} \left(\frac{a^x - 1}{x}\right) = \log a\right]$$

$$\therefore \text{k} = (\log 5)^2$$

Exercise 8.1 | Q 4.3 | Page 112

For what values of a and b is the function

f(x) = ax + 2b + 18 for $x \le 0$
$= x^2 + 3a - b$ for $0 < x \le 2$
= 8x – 2 for x > 2,
continuous for every x ?

SOLUTION

Function f is continuous for every x.

∴ Function f is continuous at x = 0 and x = 2

As f is continuous at x = 0.

$\therefore \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x)$

$\therefore \lim_{x \to 0^-} (ax + 2b + 18) = \lim_{x \to 0^+} \left(x^2 + 3a - b\right)$

$\therefore$ a(0) + 2b + 18 = $(0)^2$ + 3a - b

$\therefore$ 3a - 3b = 18

$\therefore$ a - b = 6 ....(i)

Also, Function f is continous at x = 2

$\therefore \lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x)$

$\therefore \lim_{x \to 2^-} \left(x^2 + 3a - b\right) = \lim_{x \to 2^+} (8x - 2)$

$\therefore$ $(2)^2$ + 3a - b = 8 (2) - 2

$\therefore$ 4 + 3a - b = 14

$\therefore$ 3a - b = 10 ...(ii)

Subtracting (i) from (ii), we get

2a = 4

$\therefore$ a = 2

Substituting a = 2 in (i), we get

2 - b = 6

$\therefore$ b = -4

$\therefore$ a = 2 and b = -4

Exercise 8.1 | Q 4.4 | Page 112

For what values of a and b is the function

f(x) = $\dfrac{x^2-4}{x-2}$ for x < 2

= $ax^2 - bx + 3$ for $2 \le x < 3$

= 2x – a + b for $x \ge 3$

continuous in its domain.

SOLUTION

Function f is continuous for every x on R.

∴ Function f is continuous at x = 2 and x = 3.

As f is continuous at x = 2.

$$\therefore \lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x)$$

$$\therefore \lim_{x\to 2^-} \frac{x^2-4}{x-2} = \lim_{x\to 2^+} \left(ax^2 - bx + 3\right)$$

$$\therefore \lim_{x\to 2^-} \frac{(x-2)(x+2)}{x-2} = \lim_{x\to 2^+} \left(ax^2 - bx + 3\right)$$

$$\therefore \lim_{x\to 2^-} (x+2) = \lim_{x\to 2^+} \left(ax^2 - bx + 3\right) \text{........}[\because x \to 2 \therefore x \neq 2 \therefore x-2 \neq 0]$$

∴ 2 + 2 = a$(2)^2$ - b(2) + 3

∴ 4 = 4a – 2b + 3

∴ 4a - 2b = 1 ...(i)

Also function f is continuous at x = 3

$$\therefore \lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x)$$

$$\therefore \lim_{x\to 3^-} \left(ax^2 - bx + 3\right) = \lim_{x\to 3^+} (2x - a + b)$$

$\therefore$ a(3)^2 - b(3) + 3 = 2(3) - a + b

$\therefore$ 9a - 3b + 3 = 6 - a + b

$\therefore$ 10a - 4b = 3 ....(ii)

Multiplying (i) by 2, we get

8a - 4b = 2 .....(iii)

Subtracting (iii) from (ii), we get

2a = 1

$\therefore$ a = $\frac{1}{2}$

Substituting a = $\frac{1}{2}$ in (i), we get

$4\left(\frac{1}{2}\right) - 2b = 1$

$\therefore$ 2 - 2b = 1

$\therefore$ 1 = 2b

$\therefore$ b = $\frac{1}{2}$

$\therefore$ a = $\frac{1}{2}$ and b = $\frac{1}{2}$

MISCELLANEOUS EXERCISE 8 [PAGE 113]

Miscellaneous Exercise 8 | Q 1.1 | Page 113

Discuss the continuity of the following function at the point(s) or in the interval indicated against them.

f(x) = $2x^2 - 2x + 5$ for $0 \le x < 2$

= $\frac{1 - 3x - x^2}{1 - x}$ for $2 \le x < 4$

= $\frac{7 - x^2}{x - 5}$ for $4 \le x \le 7$ on its domain.

Miscellaneous Exercise 8 | Q 1.2 | Page 113

Discuss the continuity of the following function at the point(s) or in the interval indicated against them.

$$f(x)=\frac{3^{x}+3^{-x}-2}{x^{2}} \quad \text{for } x \neq 0$$

$= (\log 3)^2$ for x = 0, at x = 0

SOLUTION

f(0) = $(\log 3)^2$ .....(given)

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{3^{x}+3^{-x}-2}{x^{2}}$$

$$= \lim_{x \to 0} \frac{3^{x}+\frac{1}{3^{x}}-2}{x^{2}}$$

$$= \lim_{x \to 0} \frac{(3^{x})^{2}+1-2(3^{x})}{x^{2}.(3)^{x}}$$

$$= \lim_{x \to 0} \frac{(3^{x}-1)^{2}}{x^{2}.(3)^{x}} \quad ...[\because a^2 - 2ab + b^2 = (a - b)^2]$$

$$= \lim_{x \to 0} \left[\left(\frac{3^{x}-1}{x}\right)^{2} \times \frac{1}{3^{x}}\right]$$

$$= \lim_{x \to 0} \left(\frac{3^{x}-1}{x}\right)^{2} \times \frac{1}{\lim_{x \to 0} 3^{x}}$$

$$= (\log 3)^{2} \times \frac{1}{3^{0}} \quad .... \left[\because \lim_{x \to 0}\left(\frac{a^{n}-1}{x}\right)=\log a\right]$$

$$= (\log 3)^{2} \times \frac{1}{1}$$

$= (\log 3)^2$

$\therefore \lim_{x \to 0} f(x) = f(0)$

$\therefore$ f is continuous at x = 0

Miscellaneous Exercise 8 | Q 1.3 | Page 113

Discuss the continuity of the following function at the point(s) or in the interval indicated against them.

$f(x) = \dfrac{5^x - e^x}{2x}$ for x ≠ 0

$= \dfrac{1}{2}(\log 5 - 1)$ for x = 0 at x = 0

SOLUTION

$f(0) = \dfrac{1}{2}(\log 5 - 1)$ ...[given]

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{5^x - e^x}{2x}$$

$$= \lim_{x \to 0} \frac{5^x - 1 - e^x + 1}{2x}$$

$$= \frac{1}{2} \lim_{x \to 0} \frac{(5^x - 1) - (e^x - 1)}{x}$$

$$= \frac{1}{2} \lim_{x \to 0} \left[\frac{(5^x - 1)}{x} - \frac{(e^x - 1)}{x}\right]$$

$$= \frac{1}{2} \left(\lim_{x \to 0} \frac{5^x - 1}{x} - \lim_{x \to 0} \frac{e^x - 1}{x}\right)$$

$$= \frac{1}{2}(\log 5 - \log e) \quad .... \left[\lim_{x \to 0} \frac{a^x - 1}{x} = \log a\right]$$

$$= \frac{1}{2}(\log 5 - 1) \quad ...[\because \log e = 1]$$

$\therefore \lim_{x \to 0} f(x) = f(0)$

$\therefore$ f is continuous at x = 0

Miscellaneous Exercise 8 | Q 1.4 | Page 113

f(x) = $\dfrac{\sqrt{x+3}-2}{x^3-1}$ for x ≠ 1

= 2 for x = 1, at x = 1.

SOLUTION

f(1) = 2 ...[given]

$$\lim_{x \to 1} f(x) = \lim_{x \to 1} \frac{\sqrt{x+3}-2}{x^3-1}$$

$$= \lim_{x \to 1} \left(\frac{\sqrt{x+3}-2}{x^3-1} \times \frac{\sqrt{x+3}+2}{\sqrt{x+3}+2} \right)$$

$$= \lim_{x \to 1} \left(\frac{x+3-4}{(x^3-1)\left(\sqrt{x+3}+2\right)} \right)$$

$$= \lim_{x \to 1} \frac{x-1}{(x-1)(x^2+x+1)\left(\sqrt{x+3}+2\right)}$$

$$= \lim_{x \to 1} \frac{1}{(x^2+x+1)\left(\sqrt{x+3}+2\right)}$$ ......[As x→1, x ≠ 1 ∴ x - 1 ≠ 0]

$$= \frac{1}{\lim_{x \to 1}(x^2+x+1) \times \lim_{x \to 1}\left(\sqrt{x+3}+2\right)}$$

$$= \frac{1}{(1^2+1+1) \times \left(\sqrt{1+3}+2\right)}$$

$$= \frac{1}{3}(2+2)$$

$= \dfrac{1}{12}$

$\therefore \lim_{x \to 1} f(x) \neq f(1)$

$\therefore$ f is discontinuous at x = 1

Miscellaneous Exercise 8 | Q 1.5 | Page 113

$f(x) = \dfrac{\log x - \log 3}{x - 3}$ for x ≠ 3

= 3 for x = 3, at x = 3.

SOLUTION

f(3) = 3 ...[given]

$\lim_{x \to 3} f(x) = \lim_{x \to 3} \dfrac{\log x - \log 3}{x - 3}$

Substitute x – 3 = h

$\therefore$ x = 3 + h,

as x → 3, h → 0

$\therefore \lim_{x \to 3} f(x) = \lim_{h \to 0} \dfrac{\log(h+3) - \log 3}{3 + h - 3}$

$= \lim_{h \to 0} \dfrac{\log\left(\frac{h+3}{3}\right)}{h}$

$= \lim_{h \to 0} \dfrac{\log\left(1 + \frac{h}{3}\right)}{\left(\frac{h}{3}\right)} \times \dfrac{1}{3}$

$= \dfrac{1}{3} \lim_{h \to 0} \dfrac{\log\left(1 + \frac{h}{3}\right)}{\left(\frac{h}{3}\right)}$

$= \dfrac{1}{3}(1)$ ..... $\left[\because \lim_{x \to 0} \dfrac{\log(1+x)}{x} = 1\right]$

$= \frac{1}{3}$

$\therefore \lim_{x \to 3} f(x) \neq f(3)$

$\therefore$ f is discontinuous at x = 3

Miscellaneous Exercise 8 | Q 2.1 | Page 113

Find k if the following function is continuous at the points indicated against them.

$f(x) = \left(\frac{5x-8}{8-3x}\right)^{\frac{3}{2x-4}}$ for x ≠ 2

= k for x = 2 at x = 2.

SOLUTION

f is continuous at x = 2

$\therefore$ f(2) = $\lim_{x \to 2} f(x)$

$\therefore$ k = $\lim_{x \to 2} \left(\frac{5x-8}{8-3x}\right)^{\frac{3}{2x-4}}$

Substitute x - 2 = h

$\therefore$ x = 2 + h

As x → 2, h → 0

$\therefore$ k = $\lim_{h \to 0} \left[\frac{5(2+h)-8}{8-3(2+h)}\right]^{\frac{3}{2(2+h)-4}}$

= $\lim_{h \to 0} \left(\frac{10+5h-8}{8-6-3h}\right)^{\frac{3}{2h}}$

= $\lim_{h \to 0} \left(\frac{2+5h}{2-3h}\right)^{\frac{3}{2h}}$

$$= \lim_{h \to 0} \frac{\left(1 + \frac{5h}{2}\right)^{\frac{3}{2h}}}{\left(1 - \frac{3h}{2}\right)^{\frac{3}{2h}}}$$

$$= \frac{\lim_{h \to 0} \left[\left(1 + \frac{5h}{2}\right)^{\frac{2}{5h}}\right]^{\frac{5}{2} \times \frac{3}{2}}}{\lim_{h \to 0} \left[\left(1 - \frac{3h}{2}\right)^{\frac{-2}{3h}}\right]^{\frac{-3}{2} \times \frac{3}{2}}}$$

$$= \frac{e^{\frac{15}{4}}}{e^{\frac{-9}{4}}} \quad \text{....} \left[\because h \to 0, \frac{5h}{2} \to 0, \frac{-3h}{2} \to 0 \text{ and } \lim_{x \to 0} (1 + x)^{\frac{1}{x}} = e\right]$$

$$= e^{\frac{24}{4}}$$

$\therefore$ k = e^6

Miscellaneous Exercise 8 | Q 2.2 | Page 113

Find k if the following function is continuous at the points indicated against them.

$$f(x) = \frac{45^x - 9^x - 5^x + 1}{(k^x - 1)(3^x - 1)} \quad \text{for } x \neq 0$$

$$= \frac{2}{3} \quad \text{for } x = 0, \text{ at } x = 0$$

SOLUTION

f is continuous at x = 0

$$\therefore \lim_{x \to 0} f(x) = f(0)$$

$$\therefore \lim_{x \to 0} \frac{(45)^x - 9^x - 5^x + 1}{(k^x - 1)(3^x - 1)} = \frac{2}{3}$$

$$\therefore \lim_{x \to 0} \frac{9^x.5^x - 9^x - 5^x + 1}{(k^x - 1)(3^x - 1)} = \frac{2}{3}$$

$$\therefore \lim_{x \to 0} \frac{9^x(5^x - 1) - 1(5^x - 1)}{(k^x - 1)(3^x - 1)} = \frac{2}{3}$$

$$\therefore \lim_{x\to 0}\frac{(5^x-1)(9^x-1)}{(k^x-1)(3^x-1)}=\frac{2}{3}$$

$$\therefore \lim_{x\to 0}\frac{\frac{(5^x-1)(9^x-1)}{x^2}}{\frac{(k^x-1)(3^x-1)}{x^2}}=\frac{2}{3}$$ ...[$\because$ x→0, $\therefore$ x ≠ 0 $\therefore$ $x^2 \neq 0$ Divide Numeratorand Denominator by x^2]

$$\therefore \frac{\lim_{x\to 0}\left(\frac{5^x-1}{x}\right)\left(\frac{9^x-1}{x}\right)}{\lim_{x\to 0}\left(\frac{k^x-1}{x}\right)\left(\frac{3^x-1}{x}\right)}=\frac{2}{3}$$

$$\therefore \frac{\log 5.\log 9}{\log k.\log 3}=\frac{2}{3}\quad\left[\because \lim_{x\to 0}\frac{a^x-1}{x}=\log a\right]$$

$$\therefore \frac{\log 5.\log(3)^2}{\log k.\log 3}=\frac{2}{3}$$

$$\therefore \frac{2\times\log 5\times\log 3}{\log k\times\log 3}=\frac{2}{3}$$

$$\therefore \frac{\log 5}{\log k}=\frac{1}{3}$$

$\therefore$ 3log 5 = log k

$\therefore$ $\log(5)^3$ = log k

$\therefore$ $(5)^3$ = k

$\therefore$ k = 125

Miscellaneous Exercise 8 | Q 2.3 | Page 113

Find k if the following function is continuous at the points indicated against them.

$f(x)=(1+kx)^{\frac{1}{x}}$, for x ≠ 0

= $e^{\frac{3}{2}}$, for x = 0, at x = 0

SOLUTION

f is continuous at x = 0

$$\therefore \lim_{x\to 0} f(x)=f(0)$$

$$\therefore \lim_{x \to 0} (1+kx)^{\frac{1}{x}} = e^{\frac{3}{2}}$$

$$\therefore \lim_{x \to 0} \left[(1+kx)^{\frac{1}{kx}}\right]^k = e^{\frac{3}{2}}$$

$$\therefore e^k = e^{\frac{3}{2}} \text{ ....} \left[\lim_{x \to 0} (1+x)^{\frac{1}{x}} = e\right]$$

$\therefore$ k = $\frac{3}{2}$

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Find a and b if the following function is continuous at the point indicated against them.

$f(x) = x^2 + a$, for x ≥ 0

= $2\sqrt{x^2+1}+b$, for x < 0 and f(1) = 2 is continuous at x = 0

SOLUTION

Since, f(x) = x^2 + a,   x ≥ 0

$\therefore$ f(1) = $(1)^2$ + a

$\therefore$ 2 = 1 + a ....[$\because$ f(1) = 2]

$\therefore$ a = 1

Also f is continuous at x = 0

$$\therefore \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x)$$

$$\therefore \lim_{x \to 0^-} \left(2\sqrt{x^2+1}+b\right) = \lim_{x \to 0^+} \left(x^2+a\right)$$

$$\therefore \left(2\sqrt{0^2+1}+b\right) = 0^2+1$$

$$\therefore 2\sqrt{0^2+1}+b = 0^2+1$$

$\therefore$ 2(1) + b = 1

$\therefore$ b = −1

$\therefore$ a = 1 and b = −1

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Find a and b if the following function is continuous at the point indicated against them.

$$f(x) = \frac{x^2 - 9}{x - 3} + a \text{ , for x > 3}$$

= 5 , x = 3

= 2x^2 + 3x + b , for x < 3

is continuous at x = 3

SOLUTION

f is continuous at x = 3

$\therefore f(3) = \lim_{x \to 3^-} f(x)$

$= \lim_{x \to 3^-} \left(2x^2 + 3x + b\right)$

$\therefore 5 = 2(3)^2 + 3(3) + b$

$\therefore 5 = 18 + 9 + b$

$\therefore b = -22$

Also, f(3) = $f(3) = \lim_{x \to 3^+} f(x)$

$\therefore 5 = \lim_{x \to 3^+} \frac{x^2 - 9}{x - 3} + a$

$= \lim_{x \to 3^+} \frac{(x + 3)(x - 3)}{(x - 3)} + a$

$= \lim_{x \to 3^+} (x + 3) + a$ ....[$\because x \to 3; x \neq 3 \therefore x - 3 \neq 0$]

= (3 + 3) + a

$\therefore 5 = 6 + a$

$\therefore a = -1$

$\therefore a = -1, b = -22$

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Find a and b if the following function is continuous at the point indicated against them.

$f(x)=\dfrac{32^x-1}{8^x-1}+a$, for x > 0

= 2 , for x = 0

= x + 5 − 2b , for x < 0

is continuous at x = 0

SOLUTION

f is continuous at x = 0

$\therefore \lim_{x\to 0^-} f(x)=f(0)$

$\therefore \lim_{x\to 0^-} (x+5-2b)=2$

∴ 0 + 5 − 2b = 2

∴ 5 − 2 = 2b

∴ 2b = 3

∴ b = $\dfrac{3}{2}$

Also $\lim_{x\to 0^+} f(x)=f(0)$

$\therefore \lim_{x\to 0^+}\left(\dfrac{32^x-1}{8^x-1}+a\right)=2$

$\therefore \lim_{x\to 0^+}\left(\dfrac{\frac{32^x-1}{x}}{\frac{8^x-1}{x}}\right)+\lim_{x\to 0^+} a=2$

$$\therefore \frac{\log 32}{\log 8} + a = 2 \text{ ....} \left[\because \lim_{x \to 0} \frac{a^x - 1}{x} = \log a\right]$$

$$\therefore \frac{\log(2)^5}{\log(2)^3} + a = 2$$

$$\therefore \frac{5 \log 2}{3 \log 2} + a = 2$$

$$\therefore \frac{5}{3} + a = 2$$

$\therefore$ a = $2 - \frac{5}{3}$

$\therefore$ a = $\frac{1}{3}$

$\therefore$ a = $\frac{1}{3}$ and b = $\frac{3}{2}$